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The interaction of waves with a row of circular cylinders

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The two-dimensional acoustics problem of the scattering of an obliquely incident plane wave by a row of equally-spaced circular cylinders is solved using multipole expansions. The method is superior to existing techniques available for this problem as it allows the far-field behaviour of the solution to be evaluated in a straightforward manner, and extensive results for the reflection and transmission coefficients are given. The problem described above has a direct analogue in the theory of water waves and this is also discussed.

1. Introduction

In this paper we will be concerned with a particular problem in the theory of diffraction gratings (i.e. arrays of equally spaced identical cylinders), namely the scattering of a plane sound or electromagnetic wave which is obliquely incident on a row of circular cylinders. An enormous literature exists on the general theory of gratings and the reader is directed to Petit (1980) and Wilcox (1984) for a detailed discussion and an extensive bibliography.

The case of normal incidence for such a grating has been considered by a number of authors over the last one hundred years and many different methods of solution have been used. Apart from approximate techniques (see, for example, Lamb 1945, p. 537; Martin & Dalrymple 1988) three substantially different approaches to the problem have been applied. The oldest of these uses separation of variables and is an extension of a method applicable to a finite array of cylinders due to Záviška (1913) and applied to the grating case by von Ignatowsky (1914). In this method the total potential for the problem, which satisfies the Helmholtz equation, is expressed as a sum of the incident plane wave plus a general scattered wave emanating from each cylinder. Owing to the periodicity of the geometry and the fact that the wave is normally incident upon the grating, the scattered waves from each cylinder are identical. Using addition theorems for Bessel functions the total potential can then be expressed in polar coordinates centred on one particular cylinder and the body boundary condition applied. This method is an efficient way of solving the problem but has the drawback that whilst information about the field near to the grating is simple to obtain, the far-field behaviour of the solution is not so easy to recover. This is due to the fact that the solution is expressed in terms of cylindrical functions whereas it is known that the far field for this problem is a sum of plane waves. Twersky (1962) reproduces this theory alongside the second approach which is a specialization to circular cylinders of a general technique (Twersky 1956) for calculating the effect of a grating in terms of the effect of the individual scatterers that make up the grating. A similar technique can be found in Jones (1986, §8.37). This method is very powerful as it is applicable to the case when the incident wave is not normal to the line of the grating (see below) and can be applied to any grating geometry. Thus Miles (1982) used it to provide an approximate formula for the low-frequency reflection and transmission coefficients when the cylinder size to spacing ratio is small and applied his results to the case where each cylinder was a thin flat plate making a constant angle with the line of the grating. The most recent method for normal incidence, due to Linton & Evans (1992) involves reducing the problem to one in a strip containing just one cylinder and then using multipole expansions, multipoles being functions which satisfy all the conditions of the problem apart from the body boundary condition, and which are singular at a point within the cylinder. The advantage of this method lies in the fact that these multipoles can be expanded simply in polar coordinates so that the body boundary condition can be applied but their far-field behaviour is that of a sum of plane waves. All these methods have their advantages but the method of Linton & Evans, while only being applicable to the circular cylindrical case, would appear to be the most powerful as it produces very simple results for all the quantities of interest including the reflection and transmission coefficients.

In his paper Twersky (1962) also considers the case of oblique incidence, using both the separation of variables technique and his own general grating formulation. This problem seems to have been considered first, using a small cylinder approximation, by Macfarlane (1946). The remarks concerning these methods for the normal incidence case discussed above are equally applicable to the oblique case. In this paper we will extend the multipole method to cover oblique incidence and again show that simple results for the reflection and transmission coefficients are obtained, allowing extensive computations to be made. No numerical results were given by Twersky and to the authors' knowledge, the only numerical results for reflection and transmission coefficients available for this problem appear in Achenbach, Lu & Kitahara (1988) who consider the fully three-dimensional problem of the scattering of a plane wave by a grating of circular cylinders by constructing a Fredholm integral equation of the second kind which is then solved numerically using the boundary element method. In their paper they present some results for the special case when the propagation vector is in a plane normal to the rods, corresponding to the two-dimensional problem we are considering here, and these results will be compared with those that we obtain.

There is a direct counterpart in the theory of water waves to two-dimensional grating problems. Thus in water of constant finite depth, the interaction of a plane surface wave with an array of vertical cylinders extending throughout the water depth can be reduced to a two-dimensional problem by factoring out the depth dependence, and the resulting two-dimensional potential must again satisfy the Helmholtz equation. Thus Spring & Monkmeyer (1974) used Záviška's method to consider the scattering by a finite array of vertical circular cylinders and a major simplification to this method was provided by Linton & Evans (1990). The water-wave problem of normal incidence on an infinite row of vertical circular cylinders was considered, using the same method, by Spring & Monkmeyer (1975) and also, using a Green function technique, by Miles (1983). The simplification provided by Linton & Evans (1990) can also be applied to this case and this will be illustrated, for arbitrary angle of incidence, in this paper. In the water-wave problem a quantity of interest is the force on the cylinder and the above method, like the multipole method, produces simple results. However, unlike the multipole method, it is not so straightforward to compute the reflection and transmission coefficients from this separation of variables solution.

The multipole formulation will be given in §2, the multipoles themselves being derived, using a method described by Thorne (1953), in the Appendix. In §3 the separation of variables solution given by Twersky (1962) will be described together with the simplification to this theory provided by Linton & Evans (1990). Aspects specific to the water-wave case will be discussed in §4, and extensive results given in §5.

2. Multipole formulation

We consider the two-dimensional problem that arises when a plane sound wave is incident upon an array of identical rigid circular cylinders of radius a in an acoustic medium. Let the (x, y)-plane be perpendicular to the generators of the cylinders so that the centres of the circles which are the cylinder cross-sections are situated at $x = 0$, $y = 2md$, $m = 0, \pm 1, \pm 2, \ldots$. Polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$ will also be used. The acoustic velocity potential, Φ, is assumed to be time-harmonic and we write

$$\Phi(x,y,t) = \mathrm{Re}\{\phi(x,y)\,\mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{2.1}$$

so that

$$(\nabla^2+k^2)\phi = 0 \quad \text{outside the cylinders,} \tag{2.2}$$

where $k = \omega/c$, c the speed of sound and $\omega/2\pi$ the frequency. A plane wave of unit amplitude making an angle θ_{I} $(0 \leqslant \theta_{\mathrm{I}} < \frac{1}{2}\pi)$ with the positive x-axis is incident upon the cylinders. Thus the incident wave is of the form

$$\phi_{\mathrm{I}} = \mathrm{e}^{\mathrm{i}\alpha x+\mathrm{i}\beta y}, \tag{2.3}$$

where

$$\alpha = k\cos\theta_{\mathrm{I}}; \quad \beta = k\sin\theta_{\mathrm{I}}. \tag{2.4}$$

As $\theta_{\mathrm{I}}\to 0$, $(\beta\to 0)$ we should recover the results for normal incidence given in Linton & Evans (1992).

Since the incident wave is periodic in the y-direction and the array of cylinders extends over the whole y-axis, we seek a scattered wave field which has the term $\exp(\mathrm{i}\beta y)$ in common with the incident wave. This together with the periodicity of the geometry implies that

$$\phi(x,y) = \mathrm{e}^{\mathrm{i}\beta y}\,\psi(x,y), \tag{2.5}$$

where ψ is periodic in y with period $2d$. It follows that we need only consider the strip $-d < y < d$ and we note that from (2.5) we can derive the two-independent conditions

$$\phi|_{y=d} = \mathrm{e}^{2\mathrm{i}\beta d}\,\phi|_{y=-d}, \tag{2.6}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=d} = \mathrm{e}^{2\mathrm{i}\beta d}\left.\frac{\partial\phi}{\partial y}\right|_{y=-d}. \tag{2.7}$$

The approach we now take is to construct multipoles $\phi_n^{(1)}$, $\phi_n^{(2)}$, symmetric and antisymmetric about the line $x = 0$, respectively. Such functions satisfy the Helmholtz equation in the strip $|y| < d$, $-\infty < x < \infty$, except at the origin where they are singular, and the periodicity conditions (2.6), (2.7). The derivation of these functions is carried out in the Appendix.

We then express the velocity potential as

$$\phi = \phi_{\mathrm{I}} + \sum_{n=0}^{\infty} a_n\phi_n^{(1)} + \sum_{n=1}^{\infty} b_n\phi_n^{(2)} \tag{2.8}$$

and determine the unknown constants a_n, b_n, by applying the boundary condition on the cylinder which is

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = 0. \tag{2.9}$$

Noting that the incident potential can be written (Abramowitz & Stegun, 1964, 9.1.41)

$$\phi_{\mathrm{I}} = \sum_{m=0}^{\infty} \epsilon_m\,\mathrm{i}^m J_m(kr)\cos m(\theta-\theta_{\mathrm{I}}), \tag{2.10}$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$, $m \geqslant 1$, and using the polar coordinate expansions of the multipoles (A 11), (A 19), (A 28) and (A 35) we can write, for $r < 2d$,

$$\phi(r,\theta) = \sum_{m=0}^{\infty} C_m(r)\cos m\theta + \sum_{m=1}^{\infty} S_m(r)\sin m\theta, \tag{2.11}$$

where

$$C_{2m}(r) = \epsilon_m(-1)^m J_{2m}(kr)\cos 2m\theta_{\mathrm{I}} + \sum_{n=0}^{\infty} a_n[\delta_{2m,n}H_n(kr) + E^{(1)}_{2m,n}J_{2m}(kr)], \tag{2.12}$$

$$C_{2m+1}(r) = 2\mathrm{i}(-1)^m J_{2m+1}(kr)\cos(2m+1)\theta_{\mathrm{I}} + \sum_{n=1}^{\infty} b_n[\delta_{2m+1,n}H_n(kr) + E^{(2)}_{2m+1,n}J_{2m+1}(kr)], \tag{2.13}$$

$$S_{2m}(r) = 2(-1)^m J_{2m}(kr)\sin 2m\theta_{\mathrm{I}} + \sum_{n=1}^{\infty} b_n[\delta_{2m,n}H_n(kr) + E^{(2)}_{2m,n}J_{2m}(kr)], \tag{2.14}$$

and

$$S_{2m+1}(r) = 2\mathrm{i}(-1)^m J_{2m+1}(kr)\sin(2m+1)\theta_{\mathrm{I}} + \sum_{n=0}^{\infty} a_n[\delta_{2m+1,n}H_n(kr) + E^{(1)}_{2m+1,n}J_{2m+1}(kr)]. \tag{2.15}$$

If we define

$$c_m = a_m H'_m(ka); \quad d_m = b_m H'_m(ka), \tag{2.16}$$

$$G^{(1)}_m = \begin{cases} -(-1)^{\frac{1}{2}m}\cos m\theta_{\mathrm{I}} & (m \text{ even}),\\ \mathrm{i}(-1)^{\frac{1}{2}(m+1)}\sin m\theta_{\mathrm{I}} & (m \text{ odd}),\end{cases} \tag{2.17}$$

$$G^{(2)}_m = \begin{cases} -(-1)^{\frac{1}{2}m}\sin m\theta_{\mathrm{I}} & (m \text{ even}),\\ \mathrm{i}(-1)^{\frac{1}{2}(m+1)}\cos m\theta_{\mathrm{I}} & (m \text{ odd}),\end{cases} \tag{2.18}$$

then the body boundary condition implies that

$$\sum_{n=0}^{\infty} c_n\left[\delta_{mn} + E^{(1)}_{m,n}\frac{J'_m(ka)}{H'_n(ka)}\right] = \epsilon_m J'_m(ka)\,G^{(1)}_m \quad (m = 0,1,\ldots), \tag{2.19}$$

$$\sum_{n=1}^{\infty} d_n\left[\delta_{mn} + E^{(2)}_{m,n}\frac{J'_m(ka)}{H'_n(ka)}\right] = 2J'_m(ka)\,G^{(2)}_m \quad (m = 1,2,\ldots). \tag{2.20}$$

Thus the c_n and the d_n each satisfy an infinite system of linear algebraic equations which can be solved by truncation. These equations can be shown to be equivalent to equations (4.7) and (4.8) of Linton & Evans (1992) in the limit $\beta \to 0$.

The scaling introduced by (2.16) leads to well-conditioned systems of equations but some of the formulae that we shall derive are better expressed in terms of the coefficients a_n and b_n. Thus in what follows we shall use both sets, always bearing (2.16) in mind.

The fact that the equations for the c_n and the d_n decouple is a consequence of the symmetry of the geometry about $x = 0$. An alternative method of solution is to decompose the problem into a symmetric and antisymmetric one, the sum of which is the solution to the problem in question. Thus if we define symmetric and antisymmetric potentials

$$\phi_{\mathrm{s}} = \tfrac{1}{2}\phi_{\mathrm{I}} + \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha x + \mathrm{i}\beta y} + \sum_{n=0}^{\infty} a_n\phi^{(1)}_n, \tag{2.21}$$

$$\phi_{\mathrm{a}} = \tfrac{1}{2}\phi_{\mathrm{I}} - \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\alpha x + \mathrm{i}\beta y} + \sum_{n=1}^{\infty} b_n\phi^{(2)}_n, \tag{2.22}$$

application of the body boundary condition, which is now only applied on the part of the boundary in $x < 0$, to each of these potentials results in the systems (2.19) and (2.20). It is then clear from (2.8) that $\phi = \phi_s + \phi_a$. Note that ϕ_s is the solution to the problem of scattering by a diffraction grating consisting of equally spaced semi-circular protrusions from a wall.

The systems of equations (2.19) and (2.20) can be used to simplify the polar coordinate expansion (2.11). After some algebra we find that

$$\phi(r,\theta) = \sum_{m=0}^{\infty} \left(\frac{H_m(kr)}{H'_m(ka)} - \frac{J_m(kr)}{J'_m(ka)}\right)\left(c_m \begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix} + d_m \begin{Bmatrix}\sin m\theta \\ \cos m\theta\end{Bmatrix}\right), \tag{2.23}$$

where the upper element of a bracketed pair is to be taken when m is even and the lower element when m is odd. Thus, using Wronskian relations for Bessel functions,

$$\phi(a,\theta) = -\frac{2\mathrm{i}}{\pi ka} \sum_{m=0}^{\infty} [J'_m(ka)]^{-1} \left(a_m \begin{Bmatrix}\cos m\theta \\ \sin m\theta\end{Bmatrix} + b_m \begin{Bmatrix}\sin m\theta \\ \cos m\theta\end{Bmatrix}\right). \tag{2.24}$$

The form of the potential as $|x| \to \infty$ can be obtained by substitution from (A 10), (A 18), (A 27) and (A 34) into (2.8). Thus as $x \to \pm\infty$

$$\phi \sim e^{\mathrm{i}\alpha x + \mathrm{i}\beta y} + \frac{1}{kd} \sum_{p=-M}^{N} t_p^{-1} \exp(\mathrm{i}\beta_p y \pm \mathrm{i}kxt_p)$$
$$\times \left\{ \sum_{n=0}^{\infty} [a_{2n} c_{2n}(t_p) \mp \mathrm{i} b_{2n+1} s_{2n+1}(t_p)] - \operatorname{sgn}(\beta_p) \sum_{n=0}^{\infty} [\mathrm{i}a_{2n+1} c_{2n+1}(t_p) \pm b_{2n} s_{2n}(t_p)] \right\}. \tag{2.25}$$

This is of the form

$$\phi \sim \begin{cases} \exp(\mathrm{i}\alpha x + \mathrm{i}\beta y) + \sum\limits_{p=-M}^{N} R_p \exp(\mathrm{i}\beta_p y - \mathrm{i}kxt_p) & (x \to -\infty), \\ \sum\limits_{p=-M}^{N} T_p \exp(\mathrm{i}\beta_p y + \mathrm{i}kxt_p) & (x \to +\infty), \end{cases} \tag{2.26}$$

where

$$R_p = \frac{1}{kdt_p} \left\{ \sum_{n=0}^{\infty} [a_{2n} c_{2n}(t_p) + \mathrm{i} b_{2n+1} s_{2n+1}(t_p)] - \operatorname{sgn}(\beta_p) \sum_{n=0}^{\infty} [\mathrm{i}a_{2n+1} c_{2n+1}(t_p) - b_{2n} s_{2n}(t_p)] \right\} \tag{2.27}$$

and

$$T_p = \delta_{0p} + \frac{1}{kdt_p} \left\{ \sum_{n=0}^{\infty} [a_{2n} c_{2n}(t_p) - \mathrm{i} b_{2n+1} s_{2n+1}(t_p)] - \operatorname{sgn}(\beta_p) \sum_{n=0}^{\infty} [\mathrm{i}a_{2n+1} c_{2n+1}(t_p) + b_{2n} s_{2n}(t_p)] \right\}. \tag{2.28}$$

Thus we see that in the far field the solution consists of the incident plane wave plus a sum of reflected and transmitted plane waves with amplitudes $|R_p|$, $|T_p|$, $p = -M, \ldots, N$, making angles $\pi - \theta_p$ and θ_p, respectively, with the positive x-axis. From (A 7)–(A 9) we see that a plane wave of the form $\exp(\mathrm{i}\beta_p y + \mathrm{i}kxt_p)$ makes an angle $\theta_p (-\frac{1}{2}\pi \leqslant \theta_p \leqslant \frac{1}{2}\pi)$ with the positive x-axis where $\cos\theta_p = t_p$, $\sin\theta_p = \beta_p/k$. Thus using (2.4) we have

$$\sin\theta_p = \sin\theta_{\mathrm{I}} + p\pi/kd \quad (p = -M, \ldots, N). \tag{2.29}$$

In particular $\theta_0 = \theta_{\mathrm{I}}$ and

$$R_0 = \frac{1}{kd\cos\theta_{\mathrm{I}}} \sum_{n=0}^{\infty} (-1)^n [a_{2n} \cos 2n\theta_{\mathrm{I}} + \mathrm{i}b_{2n+1} \cos(2n+1)\theta_{\mathrm{I}} - \mathrm{i}a_{2n+1} \sin(2n+1)\theta_{\mathrm{I}} - b_{2n} \sin 2n\theta_{\mathrm{I}}] \quad (2.30)$$

$$T_0 = 1 + \frac{1}{kd\cos\theta_{\mathrm{I}}} \sum_{n=0}^{\infty} (-1)^n [a_{2n} \cos 2n\theta_{\mathrm{I}} - \mathrm{i}b_{2n+1} \cos(2n+1)\theta_{\mathrm{I}} - \mathrm{i}a_{2n+1} \sin(2n+1)\theta_{\mathrm{I}} + b_{2n} \sin 2n\theta_{\mathrm{I}}] \quad (2.31)$$

where (A 6), (A 14), (A 24) and (A 25) have been used.

Applying Green's theorem to $\phi - \phi_{\mathrm{I}}$ and its complex conjugate (see, for example, Achenbach *et al.* 1988) leads to a result which represents the conservation of energy. It is

$$\sum_{p=-M}^{N} t_p (|R_p|^2 + |T_p|^2) = t_0. \quad (2.32)$$

It is straightforward to obtain the behaviour of R_0 and T_0 in the limit as $kd \rightarrow 0$ with a/d fixed or as $a/d \rightarrow 0$ with kd fixed. From (2.16), (2.19) and (2.20), using the fact that as $x \rightarrow 0$

$$J'_n(x) \sim \begin{cases} -\frac{1}{2}x & (n = 0), \\ 2^{-n}x^{n-1}/(n-1)! & (n \geqslant 1), \end{cases}$$

$$H'_n(x) \sim \begin{cases} 2\mathrm{i}/\pi x & (n = 0), \\ 2^n \mathrm{i}n!/\pi x^{n+1} & (n \geqslant 1), \end{cases}$$

it can be shown that in either of the above limits

$$a_0 \sim -\tfrac{1}{4}\pi\mathrm{i}(ka)^2, \quad (2.33)$$

$$a_1 \sim -\tfrac{1}{2}\pi(ka)^2 \sin\theta_{\mathrm{I}}, \quad (2.34)$$

$$b_1 \sim -\tfrac{1}{2}\pi(ka)^2 \cos\theta_{\mathrm{I}}, \quad (2.35)$$

and a_n, $b_n = O((ka)^4)$, $n \geqslant 2$. We thus obtain

$$R_0 \sim -\frac{\pi\mathrm{i}ka^2}{4d\cos\theta_{\mathrm{I}}}(1 + 2\cos 2\theta_{\mathrm{I}}), \quad (2.36)$$

$$T_0 \sim 1 + \frac{\pi\mathrm{i}ka^2}{4d\cos\theta_{\mathrm{I}}}. \quad (2.37)$$

When just one mode is present these approximations can be improved by applying the conservation of energy condition as described by Miles (1982). We then obtain the approximations

$$R_0 \approx -\frac{\mathrm{i}\xi(1 + 2\cos 2\theta_{\mathrm{I}})}{1 - \mathrm{i}\xi + 2\xi^2 \cos 2\theta_{\mathrm{I}} \cos^2\theta_{\mathrm{I}}}, \quad (2.38)$$

$$T_0 \approx \frac{1 - 2\xi^2 \cos 2\theta_{\mathrm{I}} \cos^2\theta_{\mathrm{I}}}{1 - \mathrm{i}\xi + 2\xi^2 \cos 2\theta_{\mathrm{I}} \cos^2\theta_{\mathrm{I}}}, \quad (2.39)$$

where $\xi = \pi ka^2/4d\cos\theta_{\mathrm{I}}$.

3. Separation of variables solution

An alternative method of solution to the problem under discussion is to use separation of variables as described in Twersky (1962). This procedure is an extension of a method devised by Záviška (1913) for scattering by a finite array of cylinders and subsequently presented in a water-wave context by Spring & Monkmeyer (1974). Linton & Evans (1990) greatly simplified the solution to the finite-array problem and in this section we will show that this simplification is also possible in the infinite-array case.

We introduce polar coordinates (r_m, θ_m), $m = 0, \pm 1, \pm 2, \ldots$, centred on the mth cylinder. Thus

$$x = r_m \cos\theta_m; \quad y - 2md = r_m \sin\theta_m, \tag{3.1}$$

and $r_0 \equiv r$, $\theta_0 \equiv \theta$. From (2.3) and (2.4) we see that the incident wave can be written

$$\phi_{\mathrm{I}} = I_m \exp[\mathrm{i}kr_m \cos(\theta_m - \theta_{\mathrm{I}})] \tag{3.2}$$

where the phase factor I_m is given by

$$I_m = \exp(2\mathrm{i}mkd \sin\theta_{\mathrm{I}}). \tag{3.3}$$

The total potential ϕ can be represented as the sum of the incident wave potential and a circular wave emanating from each cylinder. A general form for such a wave emanating from the mth cylinder is

$$\phi_{\mathrm{s}}^m = \sum_{n=-\infty}^{\infty} A_n^m Z_n H_n(kr_m) \exp(\mathrm{i}n\theta_m) \tag{3.4}$$

where

$$Z_n = J_n'(ka)/H_n'(ka). \tag{3.5}$$

The quantities A_n^m are unknown complex numbers which must be found by applying the boundary condition on the cylinders. Owing to the periodicity of the geometry it is clear that the only difference between the effect of the mth cylinder and that of the cylinder situated at the origin will be that due to the different phase of the incident wave at that cylinder. Thus we have

$$A_n^m = I_m A_n, \tag{3.6}$$

where we have written A_n for A_n^0. With this considerable simplification we now need only apply the body boundary condition on one of the cylinders.

We have

$$\phi = \phi_{\mathrm{I}} + \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} A_n^m Z_n H_n(kr_m) \exp(\mathrm{i}n\theta_m), \tag{3.7}$$

and this can be written in terms of r and θ using (2.10) and Graf's addition theorem for Bessel functions (Abramowitz & Stegun, 1964, 9.1.79):

$$\phi(r,\theta) = \sum_{n=-\infty}^{\infty} \{J_n(kr) \exp[\mathrm{i}n(\tfrac{1}{2}\pi - \theta + \theta_{\mathrm{I}})] + A_n Z_n H_n(kr) \exp(\mathrm{i}n\theta)\}$$
$$+ \sum_{\substack{m=-\infty \\ \neq 0}}^{\infty} I_m \sum_{n=-\infty}^{\infty} A_n Z_n \sum_{p=-\infty}^{\infty} J_p(kr) H_{n+p}(2mkd) \exp[\mathrm{i}p(\pi - \theta) - \tfrac{1}{2}\mathrm{i}\,\mathrm{sgn}(m)(n+p)\pi] \tag{3.8}$$

provided $r < 2d$. Applying the boundary condition (2.9) leads to the infinite system of equations

$$A_p + \sum_{n=-\infty}^{\infty} A_n Z_n \exp[\tfrac{1}{2}\mathrm{i}(p-n)\pi] \sum_{m=1}^{\infty} H_{n-p}(2mkd)[I_m + (-1)^{n-p} I_{-m}]$$
$$= -\exp[\mathrm{i}p(\tfrac{1}{2}\pi - \theta_{\mathrm{I}})] \quad (p = 0, \pm 1, \pm 2, \ldots). \tag{3.9}$$

This is equivalent to equation (29) in Twersky (1962). Again this system, like (2.19) and (2.20), must be solved by truncation. The sum over m is slowly convergent and must be considered carefully in any numerical computations. Methods for the accurate evaluation of sums of the form $\Sigma_{m=1}^{\infty} H_n(mx)$ have been considered by a number of previous authors, see for example Yeung & Sphaier (1989) and Thomas (1991).

We now simplify (3.8) so as to obtain a compact expression for ϕ near to the grating. We notice that the triple sum in (3.8) can be rearranged to give

$$\sum_{p=-\infty}^{\infty} J_p(kr)\exp(\mathrm{i}p\theta) \sum_{n=-\infty}^{\infty} A_n Z_n \exp[\tfrac{1}{2}\mathrm{i}(p-n)\pi] \sum_{m=1}^{\infty} H_{n-p}(2mkd)[I_m+(-1)^{n-p} I_{-m}],$$

which is simply, from (3.9),

$$\sum_{p=-\infty}^{\infty} J_p(kr)\exp(\mathrm{i}p\theta)(-\exp[\mathrm{i}p(\tfrac{1}{2}\pi-\theta_{\mathrm{I}})]-A_p).$$

We thus have, for $a < r < 2d$,

$$\phi(r,\theta) = \sum_{n=-\infty}^{\infty} A_n \exp(\mathrm{i}n\theta)[Z_n H_n(kr)-J_n(kr)] \tag{3.10}$$

and in particular

$$\phi(a,\theta) = -\frac{2\mathrm{i}}{\pi ka}\sum_{n=-\infty}^{\infty} \frac{A_n \exp(\mathrm{i}n\theta)}{H'_n(ka)}. \tag{3.11}$$

More generally we can show that if $a < r_m < 2d$ then

$$\phi(r_m,\theta_m) = I_m \sum_{n=-\infty}^{\infty} A_n \exp(\mathrm{i}n\theta_m)[Z_n H_n(kr_m)-J_n(kr_m)]. \tag{3.12}$$

These simple formulae are of precisely the same form as those which have been derived for the finite array case (equations (2.13) and (2.14) in Linton & Evans 1990), though the unknown coefficients are the solution of a different system of equations.

The reflection and transmission coefficients are not readily evaluated from this formulation, but comparison of (3.11) with (2.24) shows that the unknown coefficients A_n are related to the unknowns a_n and b_n by

$$a_0 = A_0 Z_0, \tag{3.13}$$

$$\begin{Bmatrix} a_n \\ b_n \end{Bmatrix} = (A_n+(-1)^n A_{-n})Z_n \quad (n \geqslant 1), \tag{3.14}$$

$$\begin{Bmatrix} b_n \\ a_n \end{Bmatrix} = \mathrm{i}(A_n-(-1)^n A_{-n})Z_n \quad (n \geqslant 1), \tag{3.15}$$

where the upper element of a bracketed pair is to be taken if n is even and the lower element when n is odd. Thus a_n and b_n can be evaluated and R_p and T_p computed from (2.27) and (2.28).

4. Water waves

The two-dimensional acoustics problem formulated in §2 has a direct analogue in three-dimensional water-wave theory. Thus for water-wave scattering by a row of vertical cylinders extending throughout the water depth – which is assumed to be constant everywhere – the depth dependence of the problem can be factored out. If z is taken as the vertical coordinate with $z = 0$ the undisturbed free surface and $z = -h$ the sea-bed, then the total velocity potential can be written

$$\Phi(x, y, z, t) = \mathrm{Re}\{\phi(x, y) \cosh k(z+h)\, \mathrm{e}^{-\mathrm{i}\omega t}\}, \tag{4.1}$$

where Φ is harmonic and k is the unique positive root of the dispersion relation

$$k \tanh kh = \omega^2/g, \tag{4.2}$$

where g is the acceleration due to gravity. The potential ϕ satisfies the same boundary-value problem as the function ϕ defined in (2.1) and is thus given by (2.8) with the a_n and b_n determined by solving (2.19) and (2.20). The far-field behaviour of ϕ is given by (2.25)–(2.31).

In the water-wave context another quantity of interest apart from the reflection and transmission coefficients is the exciting force on a cylinder, which is the integral of the dynamic fluid pressure ($= \mathrm{i}\rho\omega\phi$, ρ the fluid density), multiplied by an appropriate component of the normal to the body surface, over the body surface. Thus, in this case, the force on the cylinder situated at the origin in the x-direction is $\mathrm{Re}\{X \mathrm{e}^{-\mathrm{i}\omega t}\}$ where

$$X = -\mathrm{i}\rho\omega a \int_{-h}^{0} \int_{0}^{2\pi} \phi|_{r=a} \cos\theta \,\mathrm{d}\theta\, \mathrm{d}z, \tag{4.3}$$

which can be evaluated using (2.24) as

$$X = \frac{2\mathrm{i}\rho g \tanh kh}{k^2 J_1'(ka)} b_1, \tag{4.4}$$

or using (3.11) as

$$X = \frac{2\mathrm{i}\rho g \tanh kh}{k^2 H_1'(ka)} (A_1 - A_{-1}). \tag{4.5}$$

If the cylinder were in isolation then it is well known (MacCamy & Fuchs 1954) that this force would be $F \cos\theta_\mathrm{I}$ where

$$F = \frac{4\rho g \tanh kh}{k^2 H_1'(ka)}. \tag{4.6}$$

Thus, the force magnification factor in the x-direction, F_x, defined as $|X/F\cos\theta_\mathrm{I}|$ is given by

$$F_x = \left|\frac{b_1}{2Z_1 \cos\theta_\mathrm{I}}\right| = \left|\frac{A_1 - A_{-1}}{2\cos\theta_\mathrm{I}}\right|. \tag{4.7}$$

Similarly we write the exciting force in the y-direction as $\mathrm{Re}\{Y \mathrm{e}^{-\mathrm{i}\omega t}\}$ and then the force magnification factor in the y-direction is

$$F_y \equiv \left|\frac{Y}{F \sin\theta_\mathrm{I}}\right| = \left|\frac{a_1}{2Z_1 \sin\theta_\mathrm{I}}\right| = \left|\frac{A_1 + A_{-1}}{2\sin\theta_\mathrm{I}}\right|. \tag{4.8}$$

5. Results

The analysis of §§2 and 3 provides two alternative methods for computing the solution. First, the systems of equations (2.19) and (2.20) can be solved and then the reflection and transmission coefficients computed from (2.27) and (2.28). Secondly, we can solve (3.9) and then evaluate the coefficients a_n and b_n required for the reflection

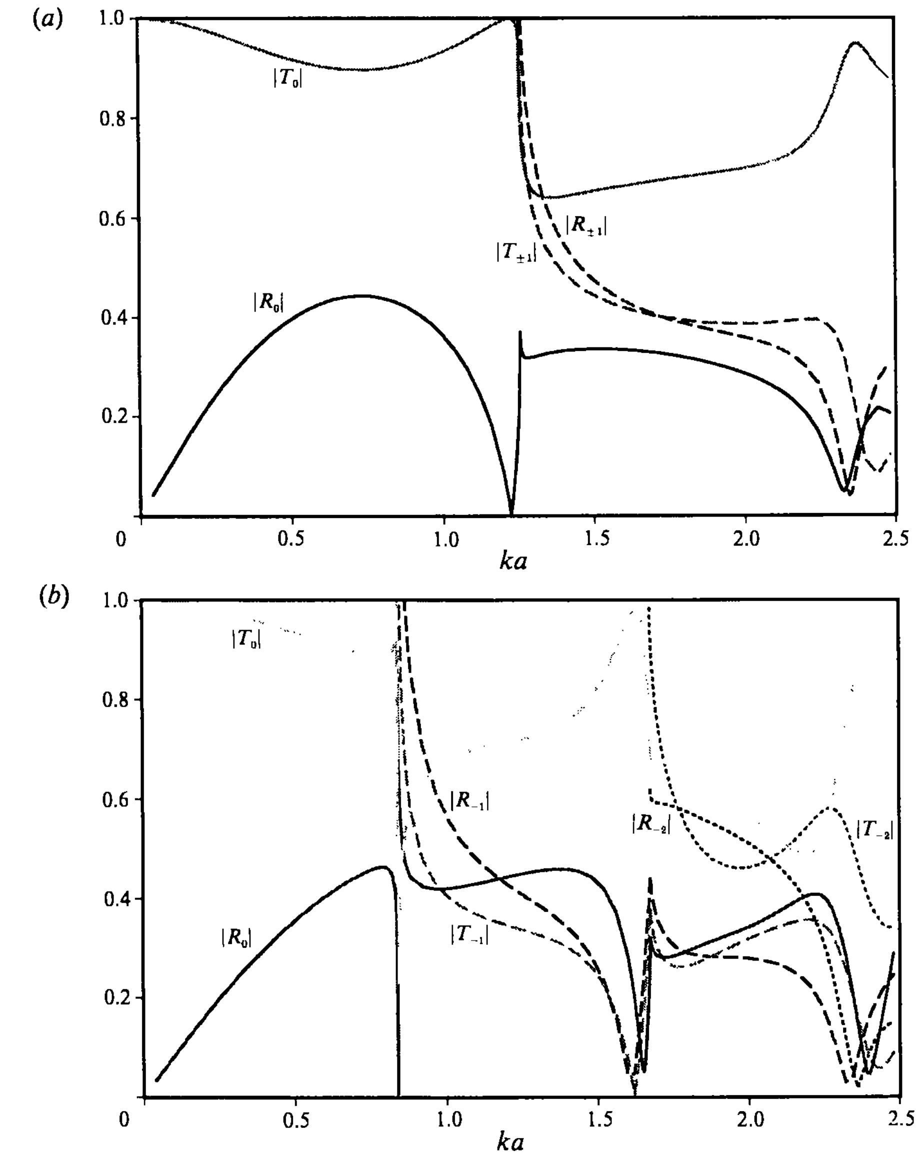

FIGURE 1 (a, b) For caption see facing page.

and transmission coefficients from (3.13)–(3.15). In both cases results can be checked against (2.32). Computations suggest that the former of these procedures is both more efficient and more accurate. For small values of a/d the two methods produce identical answers that satisfy (2.32) to high accuracy. As a/d increases, computing accurate results from (3.9) becomes more and more difficult. This problem also arises when using the multipole method though to a much lesser extent and this will be discussed below. With the above comments in mind we will restrict our attention to the use of the multipole method in what follows.

The systems of equations (2.19) and (2.20) must be solved by truncating each system to an $n \times n$ system and checking the convergence of the results as n increases. Except when a/d is very close to one, these systems converge extremely rapidly and in the results that follow a value of $n = 4$ was taken. Linton & Evans (1992) found that for

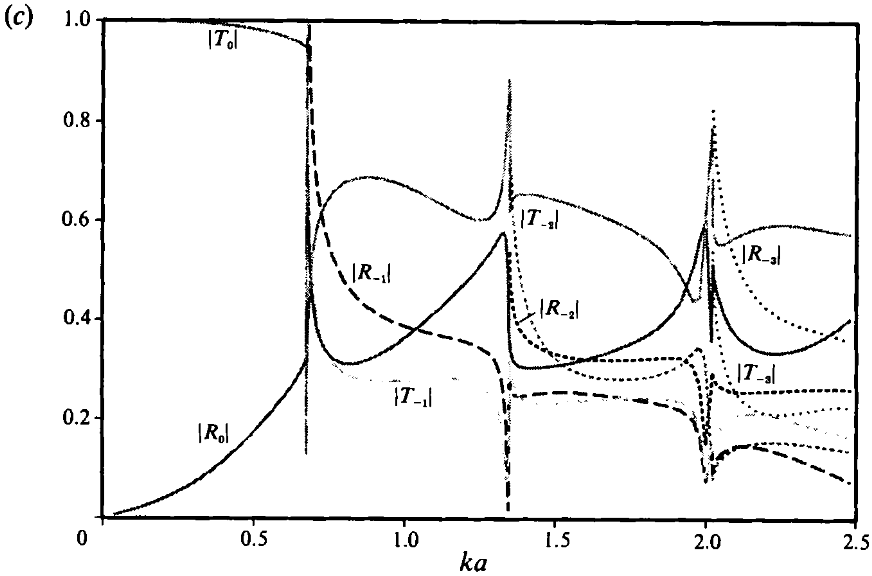

FIGURE 1. Reflected and transmitted mode amplitudes plotted against ka for a grating with $a/d = 0.4$. (*a*) $\theta_I = 0°$, (*b*) $\theta_I = 30°$, (*c*) $\theta_I = 60°$.

the $\beta = 0$ problem it was sufficient to truncate their equations to 2×2 systems and this is in fact equivalent to using $n = 4$ in the oblique incidence case since when $\beta = 0$ only the even coefficients in (2.19) and the odd coefficients in (2.20) are non-zero. All the results that are presented below have been checked against the formula (2.32) and in all cases the difference between the left- and right-hand sides of this equation was less than 10^{-4}.

We begin by comparing our results with those of Achenbach *et al.* (1988). In their paper, which considers the fully three-dimensional problem of scattering by a grating of circular cylinders using a numerical method, they present some results for the special case corresponding to the two-dimensional problem under consideration in this paper. Thus their figures 4(*a*)–(*c*) show the absolute values of the reflection and transmission coefficients plotted against ka when $a/d = 0.4$ for the three cases $\theta_I = 0°$, 30° and 60°. Our results for these cases, computed using the multipole method, are shown in figure 1. In general, agreement is good, though there are some discrepancies, particularly, in the $\theta_I = 60°$ case. Our results, being much simpler to evaluate, have been computed at very small intervals of ka so as to resolve the very spiky nature of the solution, whereas the results in Achenbach *et al.* (1988) seem not to have been evaluated at so many points.

The number of propagating modes that exist for a particular combination of parameters is determined by the number of integers p for which t_p, defined by (A 7), is real. Thus there is a reflected and a transmitted wave for each p which satisfies

$$-1 < \sin\theta_I + p\pi/kd < 1. \tag{5.1}$$

Achenbach *et al.* (1988) state that the reflection and transmission coefficient curves are discontinuous at the points where additional modes appear since the energy is redistributed over more propagating modes. Our work, and also results from Linton & Evans (1992), suggests that this is not the case. Whilst $|R_p|$ and $|T_p|$, $p \neq 0$, have finite values when they appear, the energy associated with such modes is $t_p|R_p|^2$ and $t_p|T_p|^2$, respectively, and careful computation of these energies just above the values of k at which the modes appear shows that the energies increase from zero (albeit very rapidly) and thus that the curves representing the modulus of the reflection and transmission coefficients are in fact continuous. An illustration is provided by figure 2 which shows the energies associated with the various modes for the case shown in figure 1(*a*), around the value of k at which the modes corresponding to $p = \pm 1$ appear.

Note that for normal incidence (as in figure 1(*a*) or 2) modes appear in pairs corresponding to $\pm p$. These wave travel at angles $\pm\theta_p$ to the x-axis and can be combined to give the channel modes discussed in Linton & Evans (1992).

The variation of the reflected and transmitted mode amplitudes with θ_{I} is shown in figure 3 for the case when $a/d = 0.5$. In figure 3(*a*), $kd = 1$ and we see from (5.1) that since $kd < \frac{1}{2}\pi$ the only mode that exists corresponds to $p = 0$, no matter what the angle of incidence is. In this case we see that most of the incident energy is transmitted through the grating for incident wave angles up to about 85° with total transmission being achieved at $\theta_{\mathrm{I}} \approx 75°$. When $kd = 2$, depicted in figure 3(*b*), reference to (5.1) shows that provided $\sin\theta_{\mathrm{I}} > \frac{1}{2}\pi - 1$, ($\theta_{\mathrm{I}} > 34.8°$), $p = -1$ gives rise to a reflected and transmitted wave. In figure 3(*c*) which shows the case when $kd = 4$ many more modes are possible with $p = -2$ in the range $\frac{1}{2}\pi - 1 < \sin\theta_{\mathrm{I}} < 1$, ($34.8° < \theta_{\mathrm{I}} < 90°$), $p = -1$ and $p = 0$ in the range $0 < \sin\theta_{\mathrm{I}} < 1$, ($0° < \theta_{\mathrm{I}} < 90°$), and $p = 1$ in the range $0 < \sin\theta_{\mathrm{I}} < 1 - \frac{1}{4}\pi$, ($0° < \theta_{\mathrm{I}} < 12.4°$). In all three of these figures we can see that as θ_{I} approaches 90°, so-called grazing incidence, the amplitudes of all but the fundamental reflected mode tend to zero, whilst $|R_0| \rightarrow 1$.

It is clear from (5.1) that the body size has no bearing on the number of propagating modes that exist. Figure 4 shows the variation of reflected and transmitted mode amplitudes with a/d for a case when two reflected and transmitted modes are present, namely $kd = 2$, $\theta_{\mathrm{I}} = 45°$. The maximum value of a/d shown in the figure is 0.8. For values of a/d greater then about 0.85 (< 1) the method does not appear to converge as the truncation parameter is increased. Numerical tests suggest that as θ_{I} increases the largest value of a/d for which convergence is achieved decreases with no problems being encountered, even when $a/d = 1$, when $\theta_{\mathrm{I}} = 0$. Note that when the method does converge, the convergence is always rapid. The cause of this lack of convergence is unclear though it is apparent on physical grounds that the problem is more difficult to model when the gaps between the cylinders is very small since very large fluid motions are being compressed into very small regions. It is also worth noting that Callan, Linton & Evans (1991), who proved the existence of trapped modes for a cylinder in a channel, were only able to prove existence when a/d was strictly less than unity.

The wide-spacing approximation (2.38) is compared with the full solution in figure 5. The amplitude of the reflection coefficient is plotted against a/d for three different values of a/d for the case when $\theta = 45°$. It can be seen that the accuracy of the approximation deteriorates as kd increases, the error for $kd = 0.5$, 1 and 1.5 being about 7%, 9% and 12% respectively when $a/d = 0.2$.

We now turn our attention to the computation from the multipole method of force magnification factors for the water-wave problem, $|F_x|$ and $|F_y|$, defined by (4.7) and (4.8), respectively. Figure 6 shows how these quantities vary with kd when $a/d = 0.5$ for three values of the incident wave angle. Figure 6(*a*) shows $|F_x|$ whilst $|F_y|$ is shown in figure 6(*b*). Note that $|F_y|$ is not defined when $\theta_{\mathrm{I}} = 0$. The spikes in the curves occur at places where modes appear/disappear and it is clear that the component of the force in line with the grating is affected by the presence of the grating to a much larger extent than that perpendicular to the grating with $|F_y| > 3$ when $\theta_{\mathrm{I}} = 30°$, $kd = 4.25$.

Figure 7 shows the variation of $|F_x|$ and $|F_y|$, respectively, with θ_{I} when $a/d = 0.5$ for the three cases $kd = 1$, 2 and 4. The reflection and transmission coefficients for these

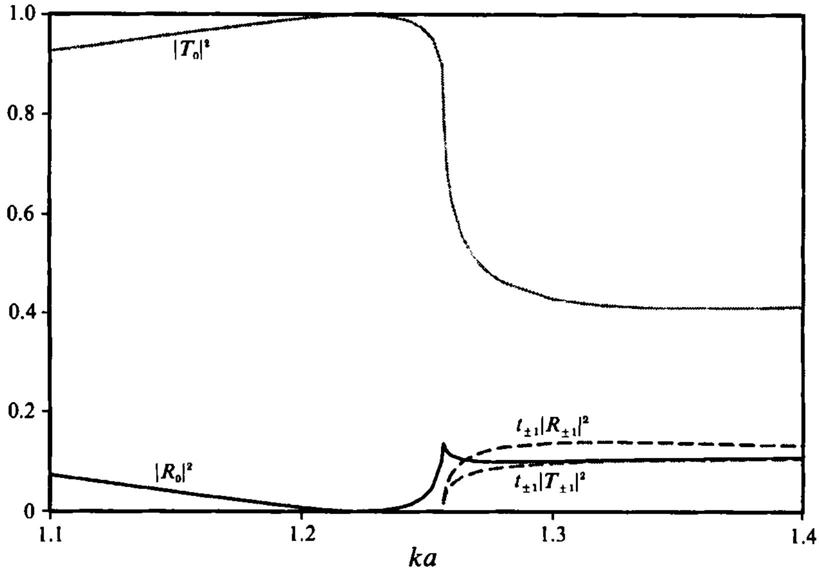

FIGURE 2. Energies associated with the various modes shown in figure 1(*a*).

parameter values were shown in figure 3. The correspondence between the spikes in the force magnification factor curves and the changes in the number of propagating modes is clear and again very large values of $|F_y|$ are apparent with $|F_y| \approx 5$ when $kd = 4$, $\theta_I = 14°$.

Finally in figure 8, $|F_x|$ and $|F_y|$ are plotted against a/d for the case $kd = 2$, $\theta_I = 45°$. The number of propagating modes present is not affected by the value of a/d and as result there are no spikes in these curves.

6. Conclusion

The two-dimensional acoustics problem of the scattering of an obliquely incident plane wave by a grating consisting of equally-spaced identical circular cylinders is solved by representing the solution as a multipole expansion, with the unknown coefficients in the expansion given by the solution of an infinite system of linear algebraic equations which can be solved numerically by truncation. The multipoles themselves can be expressed in terms of contour integrals and this enables the far-field behaviour of the solution to be simply evaluated in terms of the contributions from a finite number of poles. This leads to simple formulae for the reflection and transmission coefficients, quantities which are very difficult to obtain when this problem is solved using other known techniques.

One alternative method of solution is to use separation of variables and a description of this procedure is also included. Again an infinite system of linear algebraic equations must be solved. A major simplification to this theory, described for the case of a finite array of cylinders in Linton & Evans (1990), is given, and this allows solutions of the infinite systems that arise in the separation of variables method to be directly related to those from the multipole expansion procedure. Either system can then be solved, and the multipole expansion theory used to determine the reflection and transmission coefficients.

Numerical calculations using both methods have been performed but it was found that the multipole method was both more efficient and more accurate. Extensive results

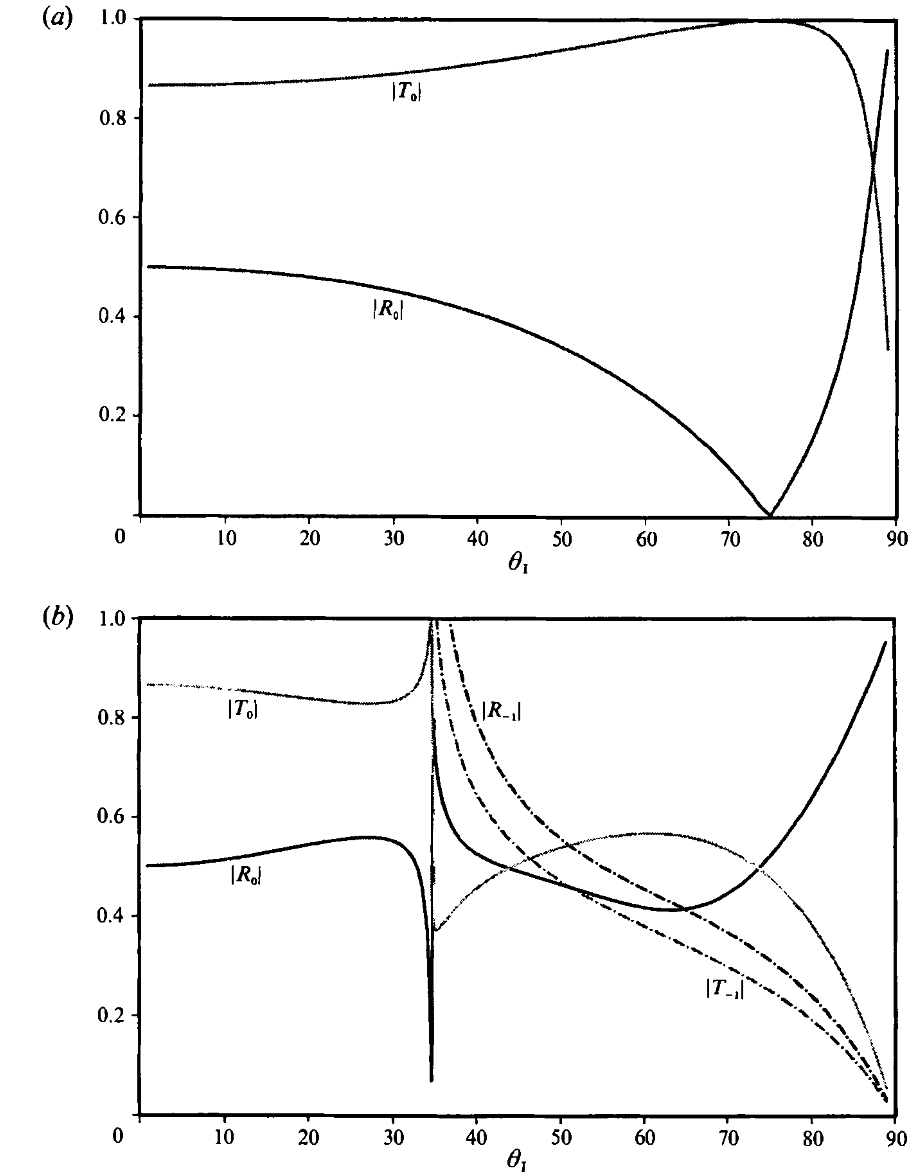

FIGURE 3(a, b) For caption see facing page.

for the reflection and transmission coefficients are presented together with force magnification factors which are relevant to the associated water-wave problem. In virtually all cases the multipole method was found to be a very accurate and highly efficient procedure. It was found however that as the angle of incidence was increased it became more difficult to obtain results for large values of a/d, i.e. values of a/d close to 1. This problem is even greater with the separation of variables technique. The reason for this lack of convergence of the infinite systems which the cylinders are nearly touching with highly oblique incidence is unclear.

(c)

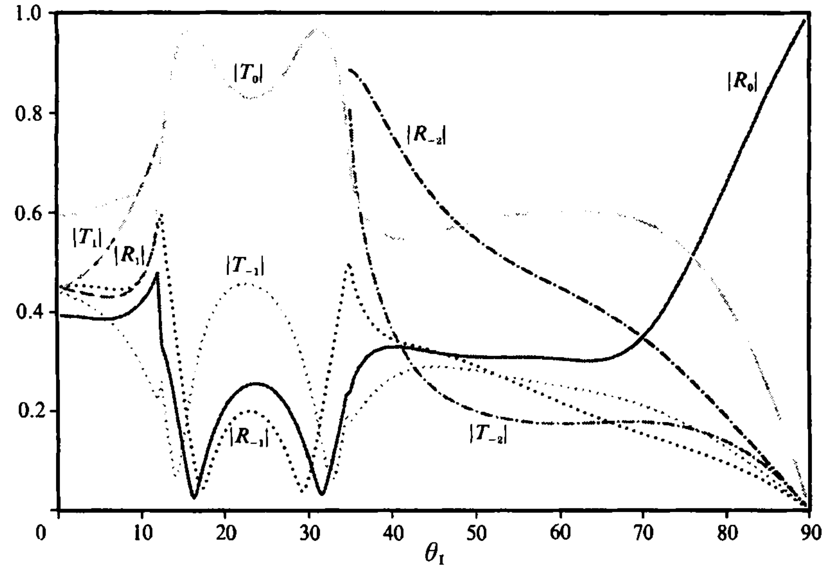

FIGURE 3. Reflected and transmitted mode amplitudes plotted against θ_I for a grating with $a/d = 0.5$. (a) $kd = 1$, (b) $kd = 2$, (c) $kd = 4$.

Appendix. Multipoles

In this Appendix we derive expressions for multipoles suitable for problems in a strip with periodic boundary conditions. Thus we seek functions ϕ which satisfy

$$(\nabla^2+k^2)\phi = 0 \quad (-\infty < x < \infty, -d < y < d), \quad \text{except at } (0,0), \tag{A 1}$$

$$\phi|_{y=d} = e^{2i\beta d}\phi|_{y=-d} \quad (\beta < k), \tag{A 2}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{y=d} = e^{2i\beta d}\left.\frac{\partial\phi}{\partial y}\right|_{y=-d}, \tag{A 3}$$

and ϕ behaves like a sum of outgoing plane waves as $|x|\to\infty$. The multipoles are constructed using the same method as was employed for channel multipoles (equivalent to the case $\beta = 0$) in Linton & Evans (1992) and only very brief details will be given here.

The method involves modifying a 'free-space' wave source $H_n(kr)\cos n\theta$ or $H_n(kr)\sin n\theta$, where $x = r\cos\theta$, $y = r\sin\theta$ and $H_n \equiv H_n^{(1)}$, to take account of the boundary conditions on $y = \pm d$. We will label the multipoles as $\phi_n^{(1)}$, $\phi_n^{(2)}$, where a superscript (1) indicates symmetry about $x = 0$ and a superscript (2) indicates antisymmetry about this line.

If we begin with $H_{2n}\cos 2n\theta$, using the integral form given by equation (2.26) in Linton & Evans (1992) suitably modified to be valid for all y, then we are led, after considerable algebra, to the result

$$\phi_{2n}^{(1)} = -\frac{2i}{\pi}\int_0^\infty \frac{e^{2i\beta d\,\mathrm{sgn}(y)}\sinh k\gamma|y| + \sinh k\gamma(2d-|y|)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)}\cos kxt\, c_{2n}(t)\,dt. \tag{A 4}$$

Here

$$\gamma(t) = \begin{cases} -i(1-t^2)^{\frac{1}{2}} & (t \leqslant 1), \\ (t^2-1)^{\frac{1}{2}} & (t > 1), \end{cases} \tag{A 5}$$

$$c_{2n}(t) = \begin{cases} \cos[2n\sin^{-1}t] & (t \leqslant 1), \\ (-1)^n\cosh[2n\cosh^{-1}t] & (t > 1), \end{cases} \tag{A 6}$$

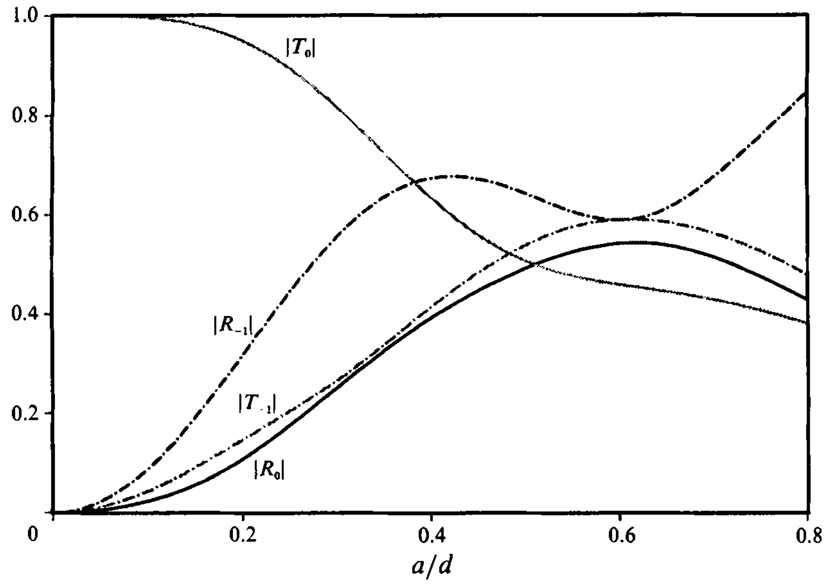

FIGURE 4. Reflected and transmitted mode amplitudes plotted against a/d for the case $kd = 2$, $\theta_I = 45°$.

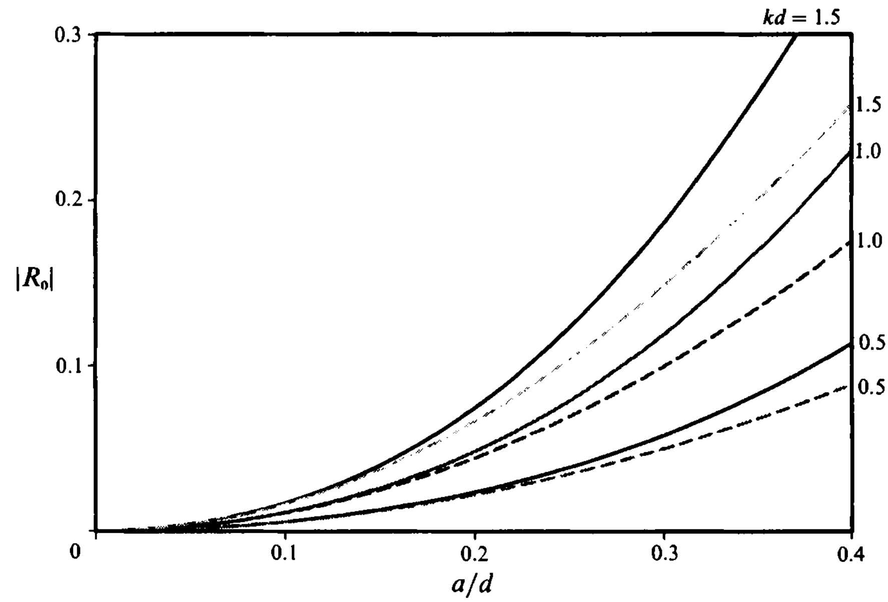

FIGURE 5. Reflected mode amplitudes plotted against a/d for the cases $kd = 0.5$, 1 and 1.5. ——, Full solution; ---, wide spacing approximation.

and the contour of integration passes beneath all the poles on the real axis. These poles are at $t = t_p$, $p = -M, \ldots, N$, where

$$t_p = [1-(\beta_p/k)^2]^{\frac{1}{2}}, \tag{A 7}$$

$$\beta_p = \beta + p\pi/d, \tag{A 8}$$

and M, N are positive integers such that

$$\beta_{-M-1} < -k < \beta_{-M}, \quad \beta_N < k < \beta_{N+1}. \tag{A 9}$$

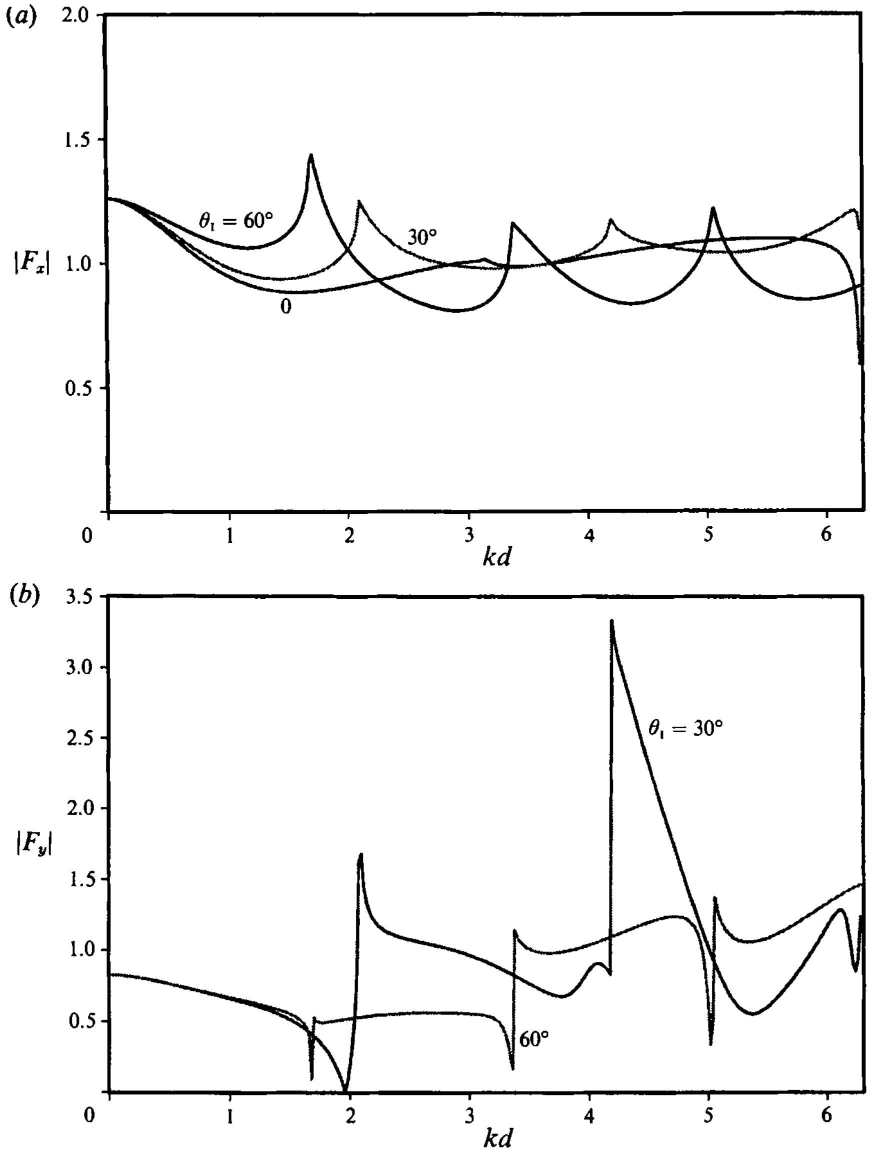

FIGURE 6. Force magnification factors plotted against a/d for a grating with $a/d = 0.5$. (*a*) x-direction, (*b*) y-direction.

It can be shown that

$$\phi_{2n}^{(1)} \sim \frac{1}{kd} \sum_{p=-M}^{N} t_p^{-1} c_{2n}(t_p) \exp(\mathrm{i}\beta_p y) \exp(\pm \mathrm{i}kxt_p) \quad \text{as } x \to \pm\infty, \tag{A 10}$$

and that

$$\phi_{2n}^{(1)}(r,\theta) = H_{2n}(kr)\cos 2n\theta + \sum_{m=0}^{\infty} [E_{2m,2n}^{(1)} J_{2m}(kr)\cos 2m\theta + E_{2m+1,2n}^{(1)} J_{2m+1}(kr)\sin(2m+1)\theta], \tag{A 11}$$

where

$$E_{2m,2n}^{(1)} = \frac{2\mathrm{i}\epsilon_m}{\pi} \int_0^\infty \frac{(\mathrm{e}^{-2k\gamma d} - \cos 2\beta d)\, c_{2n}(t)\, c_{2m}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \mathrm{d}t, \tag{A 12}$$

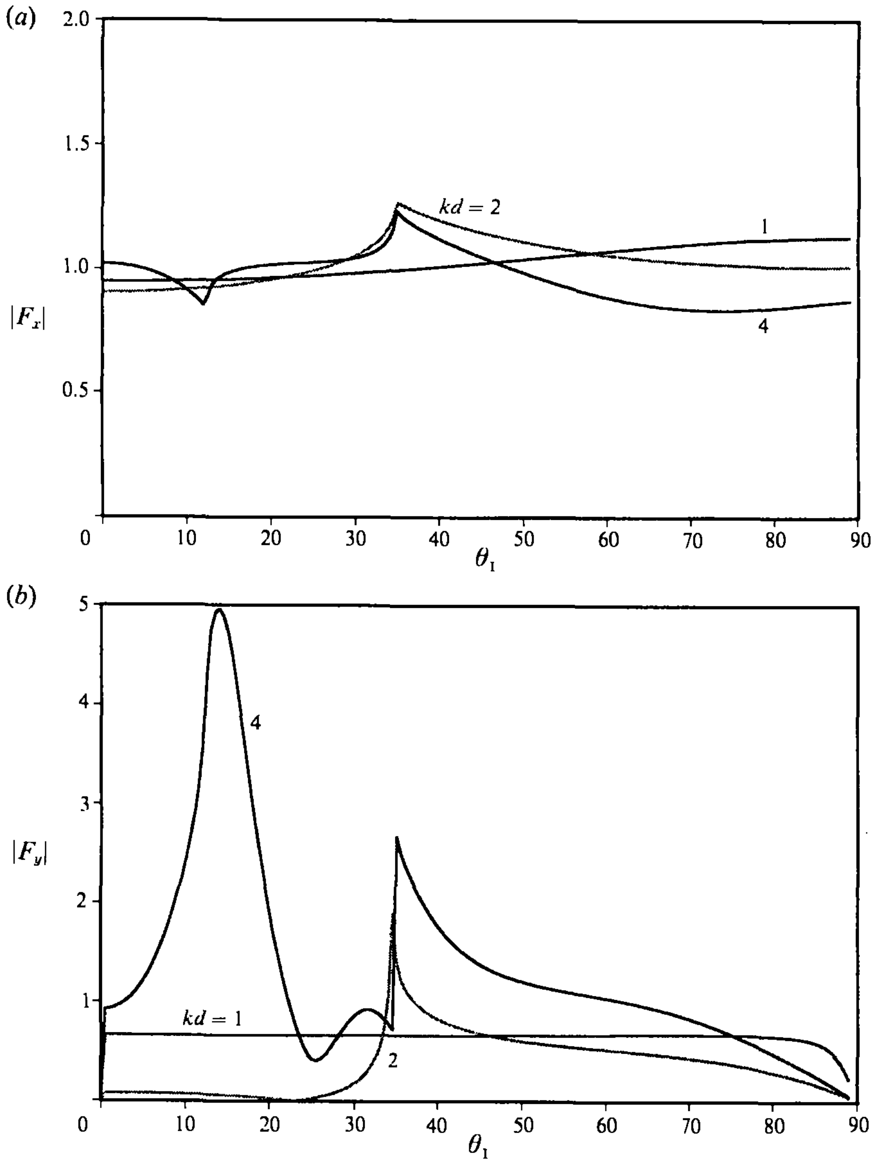

FIGURE 7. Force magnification factors plotted against θ_I for a grating with $a/d = 0.5$. (a) x-direction, (b) y-direction.

$$E^{(1)}_{2m+1,\,2n} = -\frac{4i}{\pi}\int_0^\infty \frac{\sin 2\beta d\, c_{2n}(t)\, c_{2m+1}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)}\,dt. \tag{A 13}$$

Here

$$c_{2n+1}(t) = \begin{cases} \cos[(2n+1)\sin^{-1} t] & (t \leqslant 1) \\ i(-1)^n \sinh[(2n+1)\cosh^{-1} t] & (t > 1) \end{cases}. \tag{A 14}$$

This polar coordinate expansion is valid for $r < 2d$. By splitting these integrals into a principal value integral and a sum of contributions from the poles we obtain

$$\mathrm{Re}[E^{(1)}_{2m,\,2n}] = -\delta_{mn} + \frac{\epsilon_m}{kd}\sum_{p=-M}^{N} t_p^{-1}\, c_{2n}(t_p)\, c_{2m}(t_p), \tag{A 15}$$

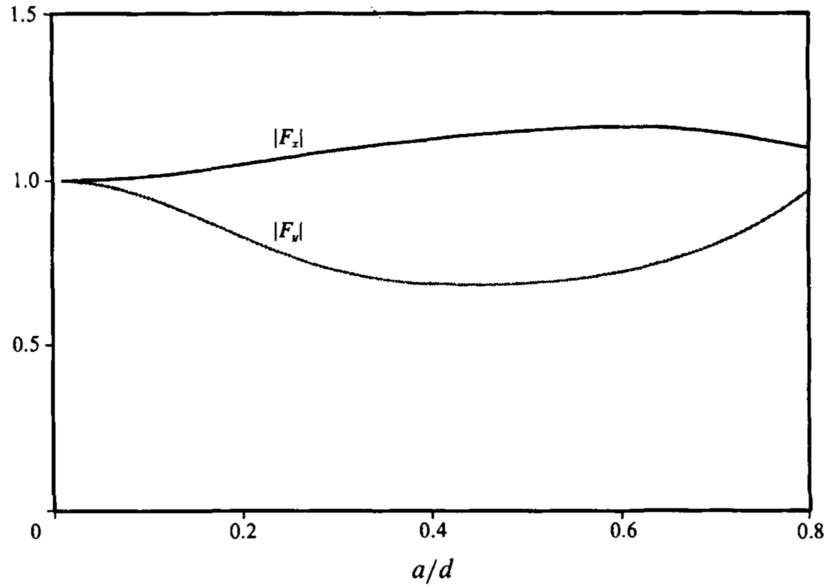

FIGURE 8. Force magnification factors plotted against a/d for the case $kd = 2$, $\theta_I = 45°$.

$$\operatorname{Im}[E^{(1)}_{2m+1,2n}] = \frac{2}{kd}\sum_{p=-M}^{N} \operatorname{sgn}(\beta_p)\, t_p^{-1}\, c_{2n}(t_p)\, c_{2m+1}(t_p). \tag{A 16}$$

The quantities $\operatorname{Im}[E^{(1)}_{2m,2n}]$ and $\operatorname{Re}[E^{(1)}_{2m+1,2n}]$ must be evaluated numerically. This can be done using the method described in Linton & Evans (1992) or that used by McIver & Bennett (1992). As $\beta \to 0$, $E^{(1)}_{2m+1,2n} \to 0$ and $\phi^{(1)}_{2n}$ tends to the channel multipole ϕ^{s}_{2n} of Linton & Evans (1992).

Similarly we have

$$\phi^{(1)}_{2n+1} = \frac{2\operatorname{sgn}(y)}{\pi}\int_0^\infty \frac{\exp[2\mathrm{i}\beta d \operatorname{sgn}(y)]\cosh k\gamma y - \cosh k\gamma(2d-|y|)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)}\cos kxt\, c_{2n+1}(t)\,\mathrm{d}t \tag{A 17}$$

$$\phi^{(1)}_{2n+1} \sim -\frac{\mathrm{i}}{kd}\sum_{p=-M}^{N} \operatorname{sgn}(\beta_p)\, t_p^{-1} c_{2n+1}(t_p)\exp(\mathrm{i}\beta_p y)\exp(\pm \mathrm{i}kxt_p) \quad \text{as } x \to \pm\infty \tag{A 18}$$

and

$$\phi^{(1)}_{2n+1}(r,\theta) = H_{2n+1}(kr)\sin(2n+1)\theta + \sum_{m=0}^{\infty}[E^{(1)}_{2m,2n+1} J_{2m}(kr)\cos 2m\theta + E^{(1)}_{2m+1,2n+1} J_{2m+1}(kr)\sin(2m+1)\theta], \tag{A 19}$$

where

$$E^{(1)}_{2m,2n+1} = \frac{2\mathrm{i}\epsilon_m}{\pi}\int_0^\infty \frac{\sin 2\beta d\, c_{2n+1}(t)\, c_{2m}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)}\mathrm{d}t, \tag{A 20}$$

$$E^{(1)}_{2m+1,2n+1} = \frac{4\mathrm{i}}{\pi}\int_0^\infty \frac{(\mathrm{e}^{-2k\gamma d} - \cos 2\beta d)\, c_{2n+1}(t)\, c_{2m+1}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)}\mathrm{d}t, \tag{A 21}$$

and

$$\operatorname{Im}[E^{(1)}_{2m,2n+1}] = -\frac{\epsilon_m}{kd}\sum_{p=-M}^{N} \operatorname{sgn}(\beta_p)\, t_p^{-1}\, c_{2n+1}(t_p)\, c_{2m}(t_p), \tag{A 22}$$

$$\operatorname{Re}[E^{(1)}_{2m+1,2n+1}] = -\delta_{mn} + \frac{2}{kd}\sum_{p=-M}^{N} t_p^{-1} c_{2n+1}(t_p)\, c_{2m+1}(t_p). \tag{A 23}$$

For multipoles antisymmetric about $x = 0$ we define

$$s_{2n}(t) = \begin{cases} \sin[2n \sin^{-1} t] & (t \leqslant 1), \\ -\mathrm{i}(-1)^n \sinh[2n \cosh^{-1} t] & (t > 1), \end{cases} \tag{A 24}$$

$$s_{2n+1}(t) = \begin{cases} \sin[(2n+1) \sin^{-1} t] & (t \leqslant 1), \\ (-1)^n \cosh[(2n+1) \cosh^{-1} t] & (t > 1), \end{cases} \tag{A 25}$$

and then

$$\phi_{2n}^{(2)} = \frac{2\,\mathrm{sgn}\,(y)}{\pi} \int_0^\infty \frac{\exp[2\mathrm{i}\beta d\,\mathrm{sgn}\,(y)] \cosh k\gamma y - \cosh k\gamma(2d-|y|)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \sin kxt\, s_{2n}(t)\, \mathrm{d}t \tag{A 26}$$

$$\phi_{2n}^{(2)} \sim \mp \frac{1}{kd} \sum_{p=-M}^{N} \mathrm{sgn}\,(\beta_p)\, t_p^{-1} s_{2n}(t_p) \exp(\mathrm{i}\beta_p y) \exp(\pm \mathrm{i}kxt_p) \quad \text{as } x \to \pm\infty, \tag{A 27}$$

$$\phi_{2n}^{(2)}(r,\theta) = H_{2n}(kr) \sin 2n\theta + \sum_{m=0}^{\infty} [E_{2m,2n}^{(2)} J_{2m}(kr) \sin 2m\theta + E_{2m+1,2n}^{(2)} J_{2m+1}(kr) \cos(2m+1)\theta], \tag{A 28}$$

where
$$E_{2m,2n}^{(2)} = \frac{4\mathrm{i}}{\pi} \int_0^\infty \frac{(\mathrm{e}^{-2k\gamma d} - \cos 2\beta d)\, s_{2n}(t)\, s_{2m}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \mathrm{d}t, \tag{A 29}$$

$$E_{2m+1,2n}^{(2)} = \frac{4\mathrm{i}}{\pi} \int_0^\infty \frac{\sin 2\beta d\, s_{2n}(t)\, s_{2m+1}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \mathrm{d}t, \tag{A 30}$$

and
$$\mathrm{Re}\,[E_{2m,2n}^{(2)}] = -\delta_{mn} + \frac{2}{kd} \sum_{p=-M}^{N} t_p^{-1} s_{2n}(t_p)\, s_{2m}(t_p), \tag{A 31}$$

$$\mathrm{Im}\,[E_{2m+1,2n}^{(2)}] = -\frac{2}{kd} \sum_{p=-M}^{N} \mathrm{sgn}\,(\beta_p)\, t_p^{-1} s_{2n}(t_p)\, s_{2m+1}(t_p). \tag{A 32}$$

Similarly

$$\phi_{2n+1}^{(2)} = -\frac{2\mathrm{i}}{\pi} \int_0^\infty \frac{\exp[2\mathrm{i}\beta d\,\mathrm{sgn}\,(y)] \sinh k\gamma|y| + \sinh k\gamma(2d-|y|)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \sin kxt\, s_{2n+1}(t)\, \mathrm{d}t \tag{A 33}$$

$$\phi_{2n+1}^{(2)} \sim \mp \frac{\mathrm{i}}{kd} \sum_{p=-M}^{N} t_p^{-1} s_{2n+1}(t_p) \exp(\mathrm{i}\beta_p y) \exp(\pm \mathrm{i}kxt_p) \quad \text{as } x \to \pm\infty, \tag{A 34}$$

$$\phi_{2n+1}^{(2)}(r,\theta) = H_{2n+1}(kr) \cos(2n+1)\theta + \sum_{m=0}^{\infty} [E_{2m,2n+1}^{(2)} J_{2m}(kr) \sin 2m\theta + E_{2m+1,2n+1}^{(2)} J_{2m+1}(kr) \cos(2m+1)\theta], \tag{A 35}$$

where
$$E_{2m,2n+1}^{(2)} = -\frac{4\mathrm{i}}{\pi} \int_0^\infty \frac{\sin 2\beta d\, s_{2n+1}(t)\, s_{2m}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \mathrm{d}t, \tag{A 36}$$

$$E_{2m+1,2n+1}^{(2)} = \frac{4\mathrm{i}}{\pi} \int_0^\infty \frac{(\mathrm{e}^{-2k\gamma d} - \cos 2\beta d)\, s_{2n+1}(t)\, s_{2m+1}(t)}{\gamma(\cosh 2k\gamma d - \cos 2\beta d)} \mathrm{d}t, \tag{A 37}$$

and
$$\operatorname{Im}[E^{(2)}_{2m,2n+1}] = \frac{2}{kd}\sum_{p=-M}^{N} \operatorname{sgn}(\beta_p)\, t_p^{-1} s_{2n+1}(t_p)\, s_{2m}(t_p), \tag{A 38}$$

$$\operatorname{Re}[E^{(2)}_{2m+1,2n+1}] = -\delta_{mn} + \frac{2}{kd}\sum_{p=-M}^{N} t_p^{-1} s_{2n+1}(t_p)\, s_{2m+1}(t_p). \tag{A 39}$$

As $\beta \to 0$, $E^{(2)}_{2m,2n+1} \to 0$ and $\phi^{(2)}_{2n+1}$ tends to the channel multipole ϕ^{s}_{2n+1} of Linton & Evans (1992).

The limits as $\beta \to 0$ of the multipoles $\phi^{(1)}_{2n+1}$ and $\phi^{(2)}_{2n}$ have no counterparts in Linton & Evans (1992). The multipoles that result when this limit is taken would be suitable for problems in a channel that have antisymmetry about the centreline of the channel with $\phi = 0$ on the channel walls.

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